MAX/MIN FOR FUNCTIONS OF SEVERAL VARIABLES

By Gloria grazioli
Critical points

Now we assume \( z = f(xy) \) is defined in an open disc in \( \mathbb{R}^2 \), centered at a point \( p_0 \).
Let’s assume that all partial derivatives of \( z = f(x,y) \) of first and second order exist and are continuous on this disc.

We call \( p_0 \) a critical point of \( z = f(x,y) \) if both first partial derivatives of \( z \) at \( p_0 \) are zero.

Geometrically this means that the graph \( z = f(x, y) \) has a horizontal tangent plane at the point \((p_0, f(p_0))\) in \( \mathbb{R}^2 \).
First and second derivative of a function of two variables

The first derivative at \( p_0 \) isn’t any more a number, it’s a row matrix

\[
z'(p_0) = (z'_x; z'_y)
\]

The second derivative of \( z \) at \( p_0 \) is now a 2×2 matrix

\[
z''(p_0) = \begin{pmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{pmatrix}
\]
Example

\[ z = f(x, y) \]
\[ z = ax^2 + bxy + cy^2 \text{ where } a, b, c \text{ are real constants.} \]
Then \((0,0)\) is a critical point and

\[ z''(0,0) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \]
The Second Derivative Test

Suppose $z$ is a real-valued function defined on an open disc in $\mathbb{R}^2$, centered at the point $p_0$ and that all partial derivatives through order two exist in this disc, and are continuous there. Suppose that $p_0$ is a critical point of $z$. 
Determinant of the matrix > 0

\[ z''(p_0) = \begin{pmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{pmatrix} \]

Then:
- If \( z''(p_0) < 0 \) is a strict local Maximum
- If \( z''(p_0) > 0 \) is a strict local minimum
Determinant of the matrix $<0$

$$z''(p_o) = \begin{pmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{pmatrix}$$

Saddle point
Determinant of the matrix=0

\[ z''(p_0) = \begin{pmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{pmatrix} \]

I can’t say anything
Exercises

Check the critical point for each of the following function. Use the second derivative test

1. $z=x^2+xy+y^2+x^3+y^3$
2. $z=3x^2y-5xy^2+2x+3y$